Guide to the Exercises:
L. Smith. *Linear Algebra.* Ch. 1-15

Chapter 1 is not central for our syllabus; however solving the exercises would be helpful. Exercises 9, 10, and 18 are slightly more difficult than the others.

Chapter 2. Exercises 1, 2, 7 are easy. $\boxed{3}$, $\boxed{4}$, $\boxed{5}$ and $\boxed{6}$ are **standard**. Exercise $\boxed{\boxed{8}}$ is important: For part (a) use the distributive property in one direction, and divide in two cases depending on $a = 0$ or not for the other direction.

Exercises 9, $\boxed{10}$, 12, $\boxed{13}$ have **medium level** of difficulty. For 10, you need to check A1-A8 routinely.

Exercise 11 is a **challenging** problem.

Chapter 3. Exercises 1-7 are **standard**. For 1, you need to check A1-A8 routinely. Exercises 2-7 can also be done this way; however they are **much shorter** if you prove that they are subpaces of some other vector space (using the results of Chapter 4).

For exercise $\boxed{8}$, first you need to carefully look at the definition of $Fun(S, A)$. Prove the equality in two steps by showing two inclusions.

Exercises $\boxed{9}$, 10, 11 and 13 have more advanced notation. Once the notation is understood, these exercises become standard.

Exercises $\boxed{12}$ and 14 are **medium level** exercises. Note that exercise 12 is equivalent to exercise 10 from chapter 2.

Exercises $\boxed{15}$ and 16 are **challenging**, but important, exercises.

Chapter 4. Exercises $\boxed{1}$, $\boxed{4}$, 5, 6, 9, 10, $\boxed{13}$, 14, 25, $\boxed{26}$, 27, and $\boxed{\boxed{29}}$ are **standard**. For exercises 2, 3, 22 and 23 try to give geometric answers. For exercises 7 and 8, try to describe the answer in terms of familiar properties of polynomials.

Exercises $\boxed{11}$, 12, $\boxed{16}$, 17, 18, $\boxed{\boxed{19}}$, 20, 21, $\boxed{24}$, 28, 30, 32, 33, 34, 35, and 36 are **medium level** exercises.

Excercise 15 will be standard, but only after fully understanding exercise 16 from chapter 3.

Exercises 31 and 37 are **challenging**.

Chapter 5 contains some incorrect exercises. Exercises 19 and 23 are **wrong**, don't attempt them. Exercise 17 is good, but it is missing an assumption that $E' \cap E'' = \emptyset$ in part (2). Exercise 22 also contains a **mistake**: It is only true if we add "$f, g \neq 0$".

Excercises $\boxed{1}$, $\boxed{2}$, 3, $\boxed{5}$, $\boxed{6}$, 7, 8, 11, 12, 13, 16 are **standard**.

Excercise 4 should be attempted only after solving Chapter 3, exercise 16.

Exercises $\boxed{9}$, $\boxed{10}$, 14, $\boxed{\boxed{18}}$, 20, 21, 24, $\boxed{26}$, 27 are **medium level** exercises. (The best way to solve 21 is to use ideas from chapter 6.)

Exercises $\boxed{\boxed{15}}$ and 25 are **challenging**.

Good further exercises are:

- Find a linearly independent subset of exercise 2 set H with the same span.
- Find a linearly independent subset of exercise 1 set E with the same span.

Chapter 6. Problems $\boxed{1}$, $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ are simple basic calculations.

Problems $\boxed{\boxed{8}}$ and $\boxed{14}$ go together – the subspace you construct in problem 8 is called the "complementary subspace." Problem 14 asks you to compute an example.

Problems 9 and $\boxed{\boxed{10}}$ go together – 9 is a special case of 10 (it is a warm-up for 10). Problem $\boxed{23}$ should be solved using 9.

Problem 12 is a very short but important problem.

Problem $\boxed{17}$ is easy but important, and extremely useful in computations.

Problem $\boxed{21}$ is a good example of finding coordinates and working with $\text{Fun}(S)$.

Problem $\boxed{25}$ is a beautiful and interesting problem, which is covered in great detail in section §9.2 of the book. The key is that the Φ_j behave like characteristic functions of t_j, so they must be independent. Then use 6.3.4.

Chapter 7 problems are mostly pretty simple. Problem $\boxed{4}$ is an important easy problem. Problem $\boxed{7}$ is an important idea and has a very standard proof. Problem $\boxed{9}$ is more difficult, but also a good computational example to solve. Problem $\boxed{\boxed{6}}$ is challenging mostly because of notation (Fun_+ and Fun_- are like "even" and "odd" functions).

Chapter 8 has many very good problems. Many of these problems show up again in chapter 10 written in a different way.

Problems $\boxed{1}$, $\boxed{2}$, $\boxed{4}$, $\boxed{5}$, and $\boxed{\boxed{25}}$ are good basic computational problems.

Problems $\boxed{6}$ and $\boxed{7}$ go together, as well as problems $\boxed{8}$ and $\boxed{9}$. The method used to solve these is the core of chapter 10 (see §10.1).

Problem 10 is unpleasant. It is a good exercise after you have done chapter 10. This idea is also discussed later, in chapter 17 (Hamilton's theorem §17.5-6).

Problem $\boxed{11}$ is an easy calculation – showing that the shift operator is nilpotent. This will show up again in §10.3.

Problems $\boxed{13}$, $\boxed{17}$, $\boxed{\boxed{18}}$, and $\boxed{\boxed{19}}$ go together. 13 is a simple application of 8.6.2. 17 is just 8.6.1 along with a dimension argument. 18 and 19 give a useful way to check for isomorphisms.

$T : \mathcal{V} \to \mathcal{W}$ is an isomorphism if **any** of the following are true:

- T has an inverse.
- $\ker(T) = \{\mathbf{0}\}$ and $\text{im}(T) = \mathcal{W}$
- $\dim(\mathcal{V}) = \dim(\mathcal{W})$ and either $\text{im}(T) = \mathcal{W}$ or $\ker(T) = \{\mathbf{0}\}$
- T(*basis of* $\mathcal{V}$) is a basis of $\mathcal{W}$.
- $\dim(\mathcal{V}) = \dim(\mathcal{W})$ and T(*basis of* $\mathcal{V}$) is either lin. indep. or has Span $= \mathcal{W}$.

Problem 17 is repeated with different notation in chapter 9 problem 19.

Problem $\boxed{16}$ is very easy, but fills an important hole in our previous work – it says that linear extensions are unique (see 8.5.1).

Problem $\boxed{20}$ is medium difficulty. The easiest way to solve this is to solve a related problem first:

Prove that $\dim(\mathcal{A} \oplus \mathcal{B}) = \dim(\mathcal{A}) + \dim(\mathcal{B})$.

Actually, the author could have written + instead of $\oplus$ for this problem since $\mathcal{W}'$ and $\mathcal{W}''$ are both subspaces of $\mathcal{V}$ and $\mathcal{W} \cap \mathcal{W}' = \{\mathbf{0}\}$.

Problem 21 is very interesting, but tricky. It is basically repeated in Chapter 10 problem 28. Try to show that $\ker(T^2 - \mathbf{I}) = \ker(T + \mathbf{I}) + \ker(T - \mathbf{I})$.

Problems $\boxed{26}$, $\boxed{27}$, $\boxed{28}$ are good example problems working with Fun. Problem 26 has an error – S must be linearly independent for this to be true. Problem 26 is about a property called "adjointness": It says that knowing a map of sets $S \to \mathcal{W}$ is equivalent to knowing a linear transformation of vector spaces $\mathrm{Span}(S) \to \mathcal{W}$ (via linear extension). Problem 27 says that a function space is isomorphic to its dual as long as S is finite (this is repeated again in Chapter 9 problem 10). The construction in problem 28 is important in many areas of math: It is called a "pullback".

Chapter 9 has just a few nice problems.

Problem 1 is a technically easy, but long and messy calculation.

Problem 2 is best solved using ideas from chapter 10 – it says that T is an isomorphism if $\det[T] \neq 0$ (see 10.4.1).

Problem 3 is the starting point of §10.1.

Problems 4, 5, 6 are basic calculations.

Problem 8 is problem 6 with an extra inverse calculation on the end. Do linear extension backwards.

Problems 9, 10 are about dual vector spaces. They are actually very easy if you can figure out the notation. Problem 9 shows that $\mathcal{V}^* \cong \mathcal{V}$, if $\mathcal{V}$ is finite dimensional. Problem 10 shows that $\mathcal{V}^{**} \neq \mathcal{V}$, if $\mathcal{V}$ is not finite dimensional.

Problem 13 is a wonderful example of a tricky linear transformation (it is a linear transformation).

Problem 14 gives a basis for $\mathcal{L}(\mathcal{V}, \mathcal{W})$. In chapter 10 (10.3.2) these are called the "elementary matrices".

Problems 15 and 16 go together. Problem 15 says that linear transformations to $\mathbb{R}^2$ can be made by writing two linear transformations to $\mathbb{R}$ next to each other. The more interesting problem is 16, which says that all linear transformations to $\mathbb{R}^2$ have this form. (This is incredibly simple using ideas from chapter 10, but it is also fun to prove without chapter 10.)

Problem 19 is fairly simple – it is just chapter 8 problem 17 written with different notation.

Chapter 10 has many easy problems.

Chapter 11. Problems 1 and $\boxed{2}$ are standard **easy** problems. 3, 4, 5 are also **easy**.

Problem $\boxed{6}$ appears in Chapter 14 as Proposition 14.1.1. This is used to calculate the **rank** of a matrix.

Problems 8, 9, 11, 12 are **standard** problems about matrices relative to interesting bases in $\mathbb{R}^n$.

Problem $\boxed{14}$ is a good problem about matrices for transformations between interesting vector spaces.

Sadly there are no problems which involve both interesting vector spaces and interesting bases.

Chapter 12. Problems 1, 2, $\boxed{3}$, 4 go together. Problem 2 is kind of silly; it is preparing for problem 4 which is a joke.

Problems 5 and 6 are simple problems about nilpotent transformations.

Problems 7 and 9 are simple prolems about cyclic transformations. Problem 8 is a bit trickier, but you can check the dimension of $\text{Span}\big\{(a, b, c),\ \mathbf{T}(a, b, c),\ \mathbf{T}^2(a, b, c)\big\}$.

The answers to 10 and 11 are obviously "No". The interesting part is finding counter-examples.

Problems 12, 13, $\boxed{14}$, $\boxed{15}$, 16 are cute, easy problems about involutions. These are also good problems for thinking of **eigenvalues** after doing Chapter 14.

Problem 19 is an easy problem, like 16, but now with reflections.

Problem $\boxed{20}$ is a wonderful problem for checking your understanding of matrices related to a transformation, and the answer is very beautiful. A good follow-up question is:

> "Now compute the matrix $\big[T\big]_{\mathcal{B}}^{\mathcal{B}}$ with respect to the basis:
> $\mathcal{B} = \big\{1 + x,\ x + x^2,\ \ldots,\ x^{n-1} + x^n,\ x^n\big\}$."   Haha!

Problems 21 and 22 are **easy** problems about matrices with respect to a basis. They try to confuse you by using the vector space of matrices. Don't be confused! $\text{Mat}_{2,2} \cong \mathbb{R}^4$!

Chapter 13. Problems 1, 2, 6 are unpleasantly long computations. Problem $\boxed{7}$ isn't too bad.

The only good computational problems in this section are $\boxed{\boxed{8}}$ and $\boxed{\boxed{9}}$.

Problems $\boxed{11}$ and 12 are standard simple theory problems relating row operations to matrix multiplication. The reason we care about this is that it allows easy computation of how row reduction changes the determinant of a matrix in Chapter 14.

Chapter 14. Problems 1, 2, 3, 4 are **basic computations**. Remember that you can compute determinants by either row reduction or row/column expansion!

Problem 5 is silly.

Problem 6 is a good diagonalization problem – a,b,e are easy; c is not too bad; d is the worst; f is nonsense.

Problems 7 and 30 rely on the fact that if $\mathbf{T}$ satisfies an equation, then so do the eigenvalues of $\mathbf{T}$ (see the Hamilton-Cayley Theorem: 17.5.1). These are similar to proposition 14.2.2... Actually problem 7 is proposition 14.2.5 in the book! (WTF??? Was the editor drunk?)

Problems 9, $\boxed{\boxed{11}}$, 12, 13, 16 are about what happens to eigenvalues/eigenspaces when you take $\mathbf{T}^k$.... these are all special cases of 11.

Problem 10 is very easy... A good follow-up question is:

> "How are the eigenvalues and eigenspaces of $\mathbf{T}$ and $\mathbf{T}-\mathbf{I}$ (and, more generally $\mathbf{T}-k\mathbf{I}$) related?"
>
> This is called "*Spectral Shift*"!

Problem 15 is too easy. An ugly solution uses the characteristic polynomial, but smart people will note that $\det(T) = \det\big([T]_{\mathcal{B}}^{\mathcal{B}}\big)$ for any basis $\mathcal{B}$, so the answer is obvious.

Problems 19 and 20 follow from the effect of simple row/column reduction steps on determinant. The easiest way to read off these effects is to convert row/column reduction steps to matrix multiplication like in Chapter 13, problem 11.

Chapter 15. Problems 1, 2, 3, 9, 10, 11 are **basic calculations**. You can skip finding cosine of the angle in 1-3.

Problems 4 and 6 ask you to check the definition of inner product. For problem 4 this is **easy**, except for positive definiteness ($\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$) – for this you should use max/min and 2nd derivative test from math 120. Problem 6 has an **error**: You need $w(x) > 0$ (not just non-negative).

Problems 12 and 13 are good examples of using Gram-Schmidt. Look at the proof of 15.2.7 and examples 6 and 7 in §15.2.

Problem 18 is a good problem about projection.